OUTLINE

• Relativistic Electrodynamics
  • What is it?

• Tensor Notation
  • Why use it?

• Mathematical Formulation
  • Background
  • Derivations
  • Operations

• Worked Example

• Applications

• Concluding Remarks
RELATIVISTIC ELECTRODYNAMICS

• What is it?
  • Formulation of the familiar Maxwell Equations taking into account relativistic effects

• Where is it applicable?
  • In the presence of large gravitational fields, or when the velocity at which the observed reference frame is “high enough”
    • Anywhere with non-negligible curvature in space-time
  • Ex. – Magnetic fields generated by Magnetars
    • Radiation Pressure at the surface of Black Holes
    • Circuitry in and signaling of orbiting and transiting spacecraft
    • Light’s curvature in the presence of large gravitational fields...
TENSOR NOTATION

Why use it?

• Provides a concise formulation of what would ordinarily be rather extensive equations and operations
• Allows one to focus primarily on underlying concepts as opposed to drawn out derivations
• Reduces the potential to make mistakes, as there are less operations and equations to actually solve, and the tensors and operations utilized in said notation are well defined
  • Short notation:
    • Helpful for electrodynamics
    • SUPER helpful for relativistic electrodynamics
MATHEMATICAL FORMULATION – BACKGROUND

• Background on relativity:
  • Time dilation – Moving clocks run slow
    \[ \Delta \tilde{t} = \sqrt{1 - \frac{v^2}{c^2}} \Delta t \]
  • Lorentz contraction – Moving objects are shortened
    \[ \Delta \tilde{x} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta x \]
    • Note – Dimensions perpendicular to velocity are not contracted
  • Recurrent factor
    \[ \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]
MATHEMATICAL FORMULATION – BACKGROUND

- In general, for movement in exclusively the x – direction:

\[ \ddot{t} = \gamma \left( t - \frac{v}{c^2} x \right) \]

\[ \ddot{x} = \gamma (x - vt) \]

\[ \ddot{y} = y \]

\[ \ddot{z} = z \]

\[ u = \frac{dx}{dt} \]

\[ \ddot{u} = \frac{d\ddot{x}}{dt} \]
MATHEMATICAL FORMULATION – BACKGROUND

• To describe space as a tensor in space-time, let $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$ with $\beta = \frac{v}{c}$.

• Then

$$
\begin{pmatrix}
\bar{x}^0 \\
\bar{x}^1 \\
\bar{x}^2 \\
\bar{x}^3
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}
$$

• Or more concisely put,

$$
\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu
$$

• Note

  • $\Lambda$ := Lorentz Transformation Matrix
  • $x^\mu$ := Contravariant 4 – vector
  • $x_\mu$ := Covariant 4 – vector
MATHEMATICAL FORMULATION – BACKGROUND

• A useful matrix → The Minkowski metric:

\[ \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

• Changing contravariant to covariant:

\[ x_\mu = \eta_{\mu\nu} x^\nu \]

• 4 Dimensional scalar product:

\[ x^\mu x_\mu = x_\mu x^\mu = \eta_{\mu\nu} x^\nu x^\mu \]
MATHEMATICAL FORMULATION – DERIVATION

- Where does this all fit in to electrodynamics?

Figure from Griffiths\textsuperscript{2}
Chapter 12
MATHEMATICAL FORMULATION – DERIVATION

• Necessary to transform Fields due to non-zero Lorentz forces

• Consider, too, the case of ambient fields and surface currents

• The derivation is lengthy, but the result is the following:
  For movement in exclusively the x – direction,

\[
\begin{align*}
\tilde{E}_x &= E_x \\
\tilde{E}_y &= \gamma (E_y - vB_z) \\
\tilde{E}_z &= \gamma (E_z + vB_y) \\
\tilde{B}_x &= B_x \\
\tilde{B}_y &= \gamma \left( B_y - \frac{v}{c^2} E_z \right) \\
\tilde{B}_z &= \gamma \left( B_z - \frac{v}{c^2} E_y \right)
\end{align*}
\]

• Without the use of tensors, each of these equations must be individually accounted for, properly represented and maintained when using each of Maxwell’s equations

• This is very labor intensive....
MATHEMATICAL FORMULATION – DERIVATION

• Introducing, the Field Tensor $F^{\mu\nu}$ and Dual Tensor $G^{\mu\nu}$

  $F^{\mu\nu}$ is a second rank, antisymmetric tensor
  
  • Basically, it’s a $4 \times 4$ matrix of values pertaining to $E$ and $B$.

• $G^{\mu\nu}$ is essentially the same matrix with substitutions of $\frac{E}{c} \rightarrow B$ and $B \rightarrow -\frac{E}{c}$

• Recalling the Lorentz transformation matrix, $\Lambda^{\mu}_{\alpha}$, the calculation of $F^{\mu\nu}$ becomes trivial to represent

  • Namely,

$$F^{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} E^{\lambda\sigma}$$
MATHEMATICAL FORMULATION – DERIVATION

• Explicitly...

\[
\bar{F}^{\mu \nu} = \begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & -B_z & B_y \\
-E_y/c & B_z & 0 & -B_x \\
-E_z/c & -B_y & B_x & 0
\end{pmatrix}
\begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\bar{F}^{\mu \nu} = \begin{pmatrix}
0 & E_x/c & \gamma(E_y/c - \beta B_z) & \gamma(E_z/c + \beta B_y) \\
-E_x/c & 0 & -\gamma(B_z - \beta E_y/c) & \gamma(B_y + \beta E_z/c) \\
-\gamma(E_y/c - \beta B_z) & \gamma(B_z - \beta E_y/c) & 0 & -B_x \\
-\gamma(E_z/c + \beta B_y) & -\gamma(B_y + \beta E_z/c) & B_x & 0
\end{pmatrix}
\]

• Again, \(G^{\mu \nu}\) is essentially the same as \(F^{\mu \nu}\) with substitutions of \(\frac{E}{c} \rightarrow B\) and \(B \rightarrow -\frac{E}{c}\)
With these tensors, all of electrodynamics may be represented by two singular equations, capable of taking into account relativity with what is now a simple transformation.

Maxwell’s Equations in Tensor Notation:

$$\frac{\partial F^{\mu \nu}}{\partial x^\nu} = \mu_0 J^\mu$$

$$\frac{\partial G^{\mu \nu}}{\partial x^\nu} = 0$$

Here, $J^\mu$ is the current density 4-vector, with

$$J^\mu = (c\rho, J_x, J_y, J_z)$$
MATHEMATICAL FORMULATION – APPLICATION

• Now, values ordinarily requiring systems of equations and laborious derivations to determine are capable of being represented by simple, easily managed and utilized equations.

• For Example...

• Electromagnetic Pressure:

\[ P = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \]

• Tangentiality:

\[ Q = -\frac{1}{4} F_{\mu\nu} G^{\mu\nu} \]

• The generalized Lorentz Force:

\[ K^\mu = q U_\nu F^{\mu\nu} \]

• Plus many, many more...
WORKED EXAMPLE

- One gains much needed insight in new topics by working through a specific example.
- Allow us to go through a worked example to truly show the power the tensor notation brings to the table.

- **Problem 12.54:** Show the second of Maxwell’s equations in tensor notation can be expressed in terms of the field tensor, $F_{\mu\nu}$, as follows:

$$
\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0
$$
WORKED EXAMPLE

- Recall that in tensor notation, the second of Maxwell’s equations is:

\[
\frac{\partial G^{\mu\nu}}{\partial x^v} = 0
\]

which simultaneously represents the “familiar” Maxwell equations:

\[
\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

- In order to solve the problem it must be shown that

\[
\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^v} = 0
\]

is equivalent to the two familiar Maxwell equations.
WORKED EXAMPLE

• We must consider the cases for each $\mu, \nu, \lambda \in \{0,1,2,3\}$

• 4 cases, 3 variables $\rightarrow$ 64 separate equations that must be considered ($4^3 = 64$)

• N.B. Many of the equations will be redundant
  • Some evaluations result in identical matrices up to a factor of -1

• Consider the case where 2 variables are equivalent
  • i.e. when $\mu = \nu$, our equation becomes

    \[
    \frac{\partial F_{\mu\mu}}{\partial x^\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\mu} = 0
    \]

  • Noting that $F_{\mu\mu} = 0$ and $F_{\mu\lambda} = -F_{\lambda\mu}$, this leads to the trivial solution $0 = 0$.

  • Non-trivial solutions when $\mu \neq \nu \neq \lambda \neq \mu$. 
WORKED EXAMPLE

• Non-trivial solutions → 2 cases
  1. All indices pertain to spatial components
     • equal to either 1, 2, or 3
  2. A single index is temporal in nature (= 0) and the others are spatial

• Case 1: All spatial
  • Consider specifically $\mu = 1, \nu = 2$, and $\lambda = 3$

\[
\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0
\]

\[
\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0
\]
WORKED EXAMPLE

\[ \frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0 \]

\[ \frac{\partial}{\partial z}(B_z) + \frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial y}(B_y) = 0 \]

• More explicitly we have shown that:

\[ \frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0 \rightarrow \nabla \cdot \mathbf{B} = 0 \]

• Any permutations maintaining the relationship \( \mu, \nu, \lambda \in \{1,2,3\} \) and \( \mu \neq \nu \neq \lambda \neq \mu \) yields an identical result, with the only difference being coefficients of -1 that, in the end, do not change the results.
WORKED EXAMPLE

• Case 2: 1 temporal, 2 spatial

• Consider specifically $\mu = 0, \nu = 1$, and $\lambda = 2$

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0$$

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0$$

$$\frac{\partial}{\partial y} \left( -\frac{E_x}{c} \right) + \frac{\partial}{\partial (ct)} (B_z) + \frac{\partial}{\partial x} \left( \frac{E_y}{c} \right) = 0$$

$$- \frac{\partial B_z}{\partial t} + \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) = 0$$
WORKED EXAMPLE

- Recognize $-\frac{\partial B_z}{\partial t} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) = 0$ as the z-component of the famous Maxwell equation:

$$-\frac{\partial B}{\partial t} = \nabla \times E$$

- The x and y components are obtained by altering values for $\nu$ and $\lambda$, where values of $\nu = 1$ and $\lambda = 3$ provide the y-component and $\nu = 2$ and $\lambda = 3$ the equation’s x-component.

- Permutations of the variables, where the solver decides to change either $\nu$ or $\lambda$ to remain 0, and the remaining two to vary, result again in an identical result up to a factor of -1.

- Thus:

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} \rightarrow -\frac{\partial B}{\partial t} = \nabla \times E$$
WORKED EXAMPLE

• We have now shown that:

\[
\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} \rightarrow -\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}
\]

\[
\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0 \rightarrow \nabla \cdot \mathbf{B} = 0
\]

which implies that

\[
\frac{\partial G^{\mu\nu}}{\partial x^v} = \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu}
\]

which was to be shown.
(Q. E. D.)
APPLICATIONS

• Knowledge of relativistic electrodynamics allows for the exploration of topics such as
  • Plasmas on curved space-times
    • Allow for the possibility of one to obtain indirect evidence of gravitational waves, strengthening the theories of the early universe
  • The radiation of moving charges
    • Scattering and dispersion of waves and energy in lossy media
    • Numerous other applications in the field of engineering

• Tensor notation becomes an invaluable tool to authors attempting to explain theories and designs, without losing the reader in multitudes of derivations
CONCLUDING REMARKS

• The beauty of tensors is in the simplistic representation that they offer to immensely complex real world situations

• The ability to pack copious amounts of information into occasionally single term equations, wholly representative of a scenario’s underlying physics is extremely powerful, and assistive to both authors and readers alike

• Finally, the simple notation paired with profound implications and preservation of situational generality, no doubt, also serves to appease the inner mathematician of any interested physicist or engineer.
RESOURCES


